Existence of an equilibrium in incomplete markets with discrete choices and many markets

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Abstract

We define and prove the existence of an equilibrium for Bewley-style models of heterogeneous agents in incomplete markets with discrete and continuous choices. Our sample model also features endogenous price volatility across many markets (locations) but still has a finite-dimensional state space, steady state equilibrium with stochastic prices. Our proof of existence uses Kakutani's Fixed Point Theorem and does not need the set of households that are indifferent between two discrete choices to be measure zero.
Keywords: Incomplete Markets, Discrete Choices, Heterogeneous Agents
JEL Classification: C62, D58, R13.

1. Introduction

Many models study the effects of incomplete insurance in the tradition of Aiyagari [1], Bewley [5], Imrohoruglu [11], Huggett [10] by adding discrete choices. For instance, Chambers et al. [6] looks at the decision to own or rent a home, Chang and Kim [7] looks at labor-force participation, while Kitao [14] looks at the decision to become an entrepreneur. Generically, proving existence has in the past in part reduced to guaranteeing that sets of households that are indifferent between two discrete choices are at most of measure zero. Otherwise aggregate demand and supply functions may not be continuous, a point discussed at length in elegant work by Chatterjee et al. [8] but goes at least as far back as Mas-Colell [17] and the references cited therein. Chatterjee et al. [8] also offers a path for proving existence in a Bewley economy when the choice space is entirely discrete.

In this paper, we provide a formal definition and path for proving existence in Bewley models when the choice space has both discrete and continuous choices and where sets of indifferent households can be arbitrarily large and thus aggregate demand and supply are correspondences rather than functions. We solve the problem created by the possibility of a large number of indiffer-

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ent households by showing that there always exists a way of allocating these indifferent households such that there is an equilibrium. No additional nuisance costs such as the ones used in Chatterjee et al. [8] are necessary. The proof, which uses Kakutani's Fixed Point Theorem (FPT) is readily adaptable to other Bewley-type models with discrete and continuous choices.

Our approach to define an equilibrium is closest to the "distributional" approach of Zame and Noguchi [23] and most recently Azevedo et al. [3]. As in their definitions, we use probability distributions over household choices so that two households with the same state space may still make different choices. Zame and Noguchi [23] and also Balder [4] study competitive equilibria in economies with continuums of agents and externalities but their and our approaches are similar to those in other, older literatures such as the one on Cournot-Nash equilibria in games with a continuum of agents and discontinuous payoff functions (e.g. Mas-Colell [18], Khan [13] and Rath [20]).

We illustrate the approach we use with a model of housing in local labor (endowment) markets. We situate a Bewley-type model of endowment shocks in incomplete markets in a Lucas and Prescott [16]-like island model of housing markets. Households' endowments of non-durable, non-storable consumption goods follow a stochastic process which is in part a function of the quality of the location they choose to live in. They may trade some of their endowment for durable housing on the island on which they choose to live. Exogenous stochastic variation in the quality of the local endowments will create endogenous household mobility and movements in house prices. The model (a simplified version of Halket and Vasudev [9]) has a continuum of discrete choices and markets - in this case locations. We show that there exists a "stationary" general equilibrium where the price of market-specific goods - in this case housing - is an exact, finite-dimensional function, even if the characteristics of a particular market (e.g. the distribution of wealth within the market) are stochastic. Heterogeneous agent, incomplete-market models with stochastic prices typically feature infinite dimensional state variables in the agents' decision problems, and thus afford only approximate solutions (as in, for instance, Krusell and Smith Jr. [15]). We build an economy for which there is an exact stationary equilibrium - in this case, the price of housing in a location will only depend upon a location-specific quality factor. This would allow us (as in Halket and Vasudev [9]) to computationally characterize prices and allocations without having the distributions over households within or across locations entering the households' state space. The proof that locations with the same quality have the same house prices relies on the upper hemi-continuity of the aggregate demand and supply correspondences and Kakutani's FPT in exactly the same way that the remedy for indifference over discrete choices does.

Island-specific markets means that the equilibrium definition needs island-specific market clearing conditions, such as those in Alvarez and Veracierto [2]. This leads to a complication in defining the "stationary" equilibrium: island-specific housing demand functions are random variables, even while more economy-wide demand functions (like the total demand for housing on all islands of a particular quality) are not. So, we proceed in two steps. First, in an object we call a consolidated stationary equilibrium, we show that there is a steady-state distribution over these economy-wide measures and that, for each location quality value, total housing demand equals total housing supply. For many of the Bewley-models cited above, this definition and existence proof would be sufficient. We then show that a stationary competitive equilibrium is just a consolidated stationary equilibrium where the allocation ensures that each island-specific market clears. The complete definition is the approach appropriate for stochastic island economies with local market clearing conditions.

2. Model

We consider an OLG island model of household consumption choice. There is a continuum of measure 1 of households and islands each in the economy.[2] Households are indexed by ι and islands are indexed by $\varepsilon \in \mathscr{E}$.

Time is discrete and infinite and each period in the economy corresponds to one year in the data. Households are born at age $a = 1$ and live at most to age T. In every period, the household survives to next period with probability, $\lambda : \mathrm{A} \to [0,1]$, which is a function of the age of the head, $a \in \mathrm{A} = \{1,...,T\}$. $\lambda(T) = 0$. We assume that $\lambda(a)$ is not only the probability for a particular individual of survival, but also the deterministic fraction of households that survive until age $a+1$ having already survived until age a.[3] Each period, a measure $b_1 = (1 + \sum_{\kappa=1}^{T} \prod_{a=1}^{\kappa} \lambda(a))^{-1}$ is born; so the population of households in the economy is stationary.

2.1. Technology

There are two goods in the economy: a non-durable, globally available, consumption good, $c \in \mathrm{C} \subseteq \mathbb{R}_+$ and a durable housing good. The housing good is island specific and in fixed supply, $\overline{H}$, on each island. Housing is "putty" within an island; households choose housing $h \in \mathrm{H}$, a bounded interval on $\mathbb{R}_+$.

Each period, households receive an endowment of consumptions goods according to a stochastic process, $l : \mathrm{I} \times \mathrm{J} \to \mathbb{R}_{++}$, which depends on the household's ability, and the quality of the island on which it chooses to live. A household's ability, indexed by i, follows a Markov chain with

[2]For a set $X \subseteq \mathbb{R}^n$, we assume that the standard Borel space is used in constructing measure and probability measure spaces. Judd [12] notes two technical issues with continuums of independent random numbers relating to the Law of Large Numbers and measurability. Sun [22] handles these issues using a Fubini extension framework; an extension which can be readily applied here too. That is, the statement "μ is a probability measure on X" implies that $(X, \mathscr{B}(X), \mu)$ is a probability measure space as in Sun [22].

[3]Using the exact law of large numbers from Sun [22] (see Corollary 2.9 therein). For cases with a continuum of stochastic processes, as below, the relevant exact law of large numbers is Theorem 2.17 of Sun [22].

state space $i \in \mathrm{I} \equiv \{1,...,I\}$ and transition probabilities given by the matrix $\pi_I(i'|i)$. The initial realization of a newborn household's ability is assumed to be drawn from the distribution Π_I for all households.

Each island's quality, indexed by j, follows a finite state Markov chain with state space $j \in \mathrm{J} \equiv \{1,...,J\}$ and transition probabilities given by the matrix $\pi_J(j'|j)$. Let Π_J denote the unique invariant measure associated with π_J. Abilities and qualities are i.i.d across households and islands, respectively.

2.2. *Preferences*

The household derives utility from housing and from the consumption good, which is the numeraire good. Preferences are time-separable where β is the time discount factor. The instantaneous utility function $u(c,h)$ is strictly increasing and concave, with the usual Inada conditions.

2.3. *Structure of the housing market*

A household can only consume housing on the island on which it lives. Housing is bought and sold in a Walrasian market where the unit price of housing on island ε of quality j is $p(j)$.

Housing is immovable and, for the households, indivisible; i.e. moving is costly. Any household that moves pays a transaction cost which is a proportion θ_h of the value of the house bought. Newborn households are born with no housing and therefore their initial location is unimportant (since, given the Inada conditions, they will pay the moving costs regardless). When households die, their housing is "sold" by the Walrasian auctioneer.

2.4. *Household's problem*

Each period, households choose which island to live on, ε', among other choices. In order to make such a choice, they must know the mapping of islands to their qualities, which is stochastic. Formally, let $\mathrm{J} = \{1,...,J\}$, $\mathscr{E} = [0,1]$ be the sets of island qualities and island names, respectively. Let Ψ_J be the set of functions $\mu_J : \mathrm{J} \times \mathscr{E} \to \{0,1\}$ such that (i) $\mu_J(j,\varepsilon) = 1$ if and only if $\mu_J(\tilde{j},\varepsilon) = 0$ for $j \neq \tilde{j}$ and (ii) $\int_{\mathscr{E}} \mu_J(j,\varepsilon)d\varepsilon = \Pi_J(j)$. In state μ_J, island ε has quality j if $\mu_J(j,\varepsilon) = 1$.

Definition. The state space $\tilde{\mathrm{S}} = \mathrm{A} \times \mathrm{I} \times \mathrm{J} \times \mathrm{H} \times \mathscr{E} \times \Psi_J$. A state can be written as $\tilde{s} = (a,i,j\_,h\_,\varepsilon\_,\mu_J) \in \tilde{\mathrm{S}}$. Let $\mathrm{S} = \mathrm{A} \times \mathrm{I} \times \mathrm{J} \times \mathrm{H}$ with elements $s = (a,i,j\_,h\_) \in \mathrm{S}$. The vector of house prices is $\vec{p} = (p(1),...,p(J)) \in \mathrm{P} \subset \mathbb{R}^J_{++}$.

The value function, $V : \tilde{\mathrm{S}} \times \mathrm{P} \to \mathbb{R}$ and optimal choice correspondence $\tilde{\mathrm{Y}} : \tilde{\mathrm{S}} \times \mathrm{P} \to \mathrm{C} \times \mathrm{H} \times \mathrm{J} \times \mathscr{E}$ (with choice vectors $\tilde{y} : \tilde{\mathrm{S}} \times \mathrm{P} \to \mathrm{C} \times \mathrm{H} \times \mathrm{J} \times \mathscr{E}$ as elements) are given by the solutions to (with $V(a = T+1,\cdot,\cdot,\cdot,\cdot,\cdot,\cdot) = 0)$:

$$V(\tilde{s},\vec{p}) = \sup_{\tilde{y} \in \tilde{\Gamma}(\tilde{s};\vec{p})} u(c,h) + \beta\lambda(a)E[V(\tilde{s}',\vec{p})|\tilde{s},\tilde{y}]$$

where $\tilde{\Gamma} : \tilde{\mathrm{S}} \times \mathrm{P} \rightrightarrows \mathrm{C} \times \mathrm{H} \times \mathrm{J} \times \mathscr{E}$ is given by

$$c + (1 + 1_m \theta_h) p(j) h + \leq l(i,j) + p(j\_) h\_$$
$$c \in \mathrm{C}$$
$$h \in \mathrm{H}$$
$$\mu_J(j,\varepsilon) > 0$$

$$1_m = \begin{cases} 0 & \text{if } h = h\_, \varepsilon = \varepsilon\_ \\ 1 & \text{else} \end{cases}$$

Lemma 1. *(Theorem of the Maximum). There is a solution to the household's problem such that*

1. There exists a unique V that solves the household's problem.
2. The optimal policy correspondence $\tilde{\mathrm{Y}}$ is non-empty, compact-valued and upper hemi-continuous.

Proof. $u(\cdot,\cdot)$ is continuous and $\tilde{\Gamma}$ is compact-valued and upper hemi-continuous. The proof follows Stokey et al. [21], Thm. 3.6 for Berge's maximum theorem, with the exception of the upper hemi-continuity of the optimal policy correspondences (for which they require continuity of the constraint set). Instead, let $V_0, \tilde{\mathrm{Y}}_0$ be the value function and optimal policy correspondence, respectively, for the same household's problem conditional on the household not moving ($1_m = 0$) and let $V_1, \tilde{\mathrm{Y}}_1$ be the same but for ($1_m = 1$), so that V is the upper envelope of V_0 and V_1. Berge's Theorem does apply to $V_0, \tilde{\mathrm{Y}}_0$ and $V_1, \tilde{\mathrm{Y}}_1$. Upper hemi-continuity of $\tilde{\mathrm{Y}}$ follows. □

Lemma 2. *For all $\tilde{s} = (a,i,j,h,\varepsilon,\mu_J) \in \tilde{\mathrm{S}}$ and $\vec{p} \in \mathrm{P}$, let $\tilde{\mathrm{Y}}(\tilde{s},\vec{p})$, be the solutions to the household's problem at $\tilde{s}$ given prices $\vec{p}$.*

(i) If $(\tilde{c},\tilde{h},\tilde{j},\tilde{\varepsilon}) \in \tilde{\mathrm{Y}}(\tilde{s},\vec{p})$ with $\tilde{h} \neq h$, or $\tilde{\varepsilon} \neq \varepsilon$ then $(\tilde{c},\tilde{h},\tilde{j},\hat{\varepsilon}) \in \tilde{\mathrm{Y}}(\tilde{s},\vec{p}) \ \ \forall \hat{\varepsilon} : \mu_J(\tilde{j},\hat{\varepsilon}) = 1$

(ii) $(\tilde{c},h,j,\hat{\varepsilon}) \notin \tilde{\mathrm{Y}}(\tilde{s},\vec{p}) \ \ \forall \hat{\varepsilon} \neq \varepsilon,\ \tilde{c} \in \mathrm{C}$

Proof. (i) follows from the fact that, conditional on moving, $\tilde{\varepsilon}$ appears only in the $\mu_J(\tilde{j},\tilde{\varepsilon}) = 1$ constraint. (ii) follows from the fact that the household will not pay the strictly positive moving cost to live in the same size house on an island of the same quality as where it already lives. □

3. Definition of Equilibrium

Section 3 defines and section 4 proves the existence of a stationary competitive equilibrium for the economy. Since our model has both discrete and continuous state variables the proof of existence of an equilibrium correspondingly differs from the one in Aiyagari [1]. Our proof involves a selection of state-contingent action plans in areas of indifference. In order to formalize this, we introduce mixed allocations which will serve as tie-breaking criteria[4]. Since our economy is populated by a continuum of agents, there is no aggregate uncertainty using a mixed allocation.

[4]The mixed allocations do not affect preferences. Rather they will just allocate indifferent households to one or another of the items in the set they are indifferent over.

3.1. Mixed allocations and the distribution of households

Definition. Let Ψ be the set of probability measures on $\mathrm{C} \times \mathrm{H} \times \mathrm{J}$, with elements $\psi : \mathscr{B}(\mathrm{C} \times \mathrm{H} \times \mathrm{J}) \to [0,1]$. Let Δ be the space of functions $f : \mathrm{S} \to \Psi$. Likewise let $\tilde{\Psi}$ be the set of probability measures on $\mathrm{C} \times \mathrm{H} \times \mathrm{J} \times \mathscr{E}$, with elements $\tilde{\psi} : \mathscr{B}(\mathrm{C} \times \mathrm{H} \times \mathrm{J} \times \mathscr{E}) \to [0,1]$.

A *mixed allocation*, $\tilde{\alpha} : \tilde{\mathrm{S}} \times \mathrm{P} \to \tilde{\Psi}$ specifies the probability distribution over a choice set given by $\tilde{\mathrm{Y}}(\tilde{s}, \vec{p})$.

$$\tilde{\alpha}(\tilde{s}, \vec{p}) \in \{\tilde{\psi} \in \tilde{\Psi} : supp(\tilde{\psi}) \subseteq \tilde{\mathrm{Y}}(\tilde{s}, \vec{p})\}$$

For any optimal choice correspondence $\tilde{\mathrm{Y}}$, define the correspondence Y so that $\mathrm{Y}(s, \vec{p}) = \{(\tilde{c}, \tilde{h}, \tilde{j}) : \exists \varepsilon, \tilde{\varepsilon} \in \mathscr{E}, \mu_J \in \Psi_J \ni (\tilde{c}, \tilde{h}, \tilde{j}, \tilde{\varepsilon}) \in \tilde{\mathrm{Y}}(s, \varepsilon, \mu_J, \vec{p})\}$. We say that $\tilde{\mathrm{Y}}$ *implies* Y and note that such implication is unique and that Y is also upper hemi-continuous.

Let $\tilde{\Lambda}$ be the space of mixed allocations generated by $\tilde{\mathrm{Y}}$. Likewise let Λ be the space of measurable functions $\alpha : \mathrm{S} \times \mathrm{P} \to \Psi$ such that $\alpha(s, \vec{p}) \in \{\psi \in \Psi : supp(\psi) \subseteq \mathrm{Y}(s, \vec{p})\}$. Let $\mathscr{M}$ be the space of probability distributions on S. Let $\tilde{\mathscr{M}}$ be the space of probability distributions $\tilde{\mu} : \mathscr{B}(\mathrm{S} \times \mathscr{E}) \to [0,1]$ such that (i) $d\tilde{\mu}(a, i, j, h, \varepsilon) > 0$ if and only if $d\tilde{\mu}(a, i, \tilde{j} \neq j, h, \varepsilon) = 0$ and (ii) $\Pi_J(j) = \int_{\varepsilon} 1\{\varepsilon : d\tilde{\mu}(\cdot, \cdot, j, \cdot, \varepsilon) > 0\} d\varepsilon$. Note also that any $\tilde{\mu} \in \tilde{\mathscr{M}}$ *implies* a unique μ_J (using an analogous definition).

3.2. Stationary competitive equilibrium

Definition. A *consolidated stationary equilibrium* (CSE) is a vector of strictly positive prices, $\vec{p}^*$, an optimal choice correspondence $\tilde{\mathrm{Y}}^*$ with implied Y^*, an $\alpha^* \in \Lambda$ and a probability measure $\mu^* \in \mathscr{M}$ such that:

(i) $\tilde{y}^* = (c^*, h^*, j^*, \varepsilon^*)$ solves the household's problem for each $\tilde{y}^* \in \tilde{\mathrm{Y}}^*$

(ii) Goods market clears:

$$\int_{\mathrm{S}} \int_{\mathrm{Y}^*(s,\vec{p}^*)} l(i, j^*(s, \vec{p}^*)) d\alpha^*(s, \vec{p}^*) d\mu^* = \int_{\mathrm{S}} \int_{\mathrm{Y}^*(s,\vec{p}^*)} (c^*(s, \vec{p}^*) + 1_m(s, \vec{p}^*) h^*(s, \vec{p}^*) p^*(j^*(s, \vec{p}^*)) \theta_h) d\alpha^*(s, \vec{p}^*) d\mu^*$$

(iii) For each quality value, total housing demand equals total housing supply:

$$\Pi_J(j')\overline{H} = \int_{\mathrm{S}} \int_{\mathrm{Y}^*(s,\vec{p}^*)} h^*(s, \vec{p}^*) \cdot 1\{j^*(s, \vec{p}^*) = j'\} d\alpha^*(s, \vec{p}^*) d\mu^* \qquad \forall j' \in \mathrm{J}$$

(iv) Steady-state distribution:

$$\mu^* = \Upsilon_{\vec{p}^*, \alpha^*} \mu^*$$

where $\Upsilon_{\vec{p}^*, \alpha^*}$ is the transition function generated by the optimal choice correspondence of the household, the mixed allocation, α^*, and the exogenous stochastic processes; 1_m is the moving indicator defined in the household's problem and $1\{a = b\}$ is an indicator function which equals 1 if $a = b$.

The definition of a CSE does not explicitly ensure that all housing markets clear. Instead it only requires that, at equilibrium prices, the total housing demand by households living on islands with quality j equals the total housing supply on all such islands (Condition (iii)).

Definition. A *stationary competitive equilibrium* is a $\{\vec{p}^*, \tilde{\mathrm{Y}}^*, \mathrm{Y}^*, \tilde{\alpha}^* \in \tilde{\Lambda}, \tilde{\mu}^* \in \tilde{\mathscr{M}}\ \}$ such that

(i) $\{\vec{p}^*, \tilde{\mathrm{Y}}^*, \mathrm{Y}^*, \alpha^*, \mu^*\ \}$ is a CSE where α^*, μ^* are given by:

$$\alpha^*(a,i,j,h,\vec{p}^*) = \int_{\varepsilon\in\mathscr{E}} d\tilde{\alpha}^*(a,i,j,h,\varepsilon,\mu_J,\vec{p}^*)$$
$$\mu^*(s) = \int_{\varepsilon\in\mathscr{E}} \mu_J(j,\varepsilon)d\tilde{\mu}^*(s,\varepsilon)$$

where μ_J is implied by $\tilde{\mu}^*$.

(ii) All housing markets always clear:

$$\overline{H} = \int_{\tilde{\mathrm{S}}} \int_{\tilde{\mathrm{Y}}^*(\tilde{s},\vec{p}^*)} h^*(\tilde{s},\vec{p}^*)\cdot 1\{\varepsilon^*(\tilde{s},\vec{p}^*) = \varepsilon'\}d\tilde{\alpha}^*(\tilde{s},\vec{p}^*)d\tilde{\mu}^* \qquad \forall \varepsilon' \in \mathscr{E}$$

The following lemma guarantees that, given a CSE, we can always ensure each island's housing market clears at the same equilibrium prices:

Lemma 3. *Let $\{\vec{p}^*, \tilde{\mathrm{Y}}^*, \mathrm{Y}^*, \alpha^*, \mu^*\ \}$ be a CSE. Then for any $\tilde{\mu} \in \tilde{\mathscr{M}}$ such that*

$$\mu^*(s) = \int_{\varepsilon\in\mathscr{E}} \mu_J(j,\varepsilon)\tilde{\mu}(s,\varepsilon)d\varepsilon$$

where μ_J is implied by $\tilde{\mu}$, there exists an $\tilde{\alpha} \in \tilde{\Lambda}$ generated by $\tilde{\mathrm{Y}}^*$ such that $\{\vec{p}^*, \tilde{\mathrm{Y}}^*, \mathrm{Y}^*, \tilde{\alpha}, \tilde{\mu}\ \}$ is a stationary competitive equilibrium.

4. Existence of Equilibrium

We use Kakutani's FPT in order to establish the existence of a stationary CE and prove lemma 3. The proof can be broadly divided into three steps:

1. The optimal policy function generates a transition function for the household distribution over states. We show that there is a household distribution over states that is invariant with respect to the transition function.
2. Show that the set of stationary household distributions over states is upper hemi-continuous in the price vector.
3. Construct a price transition operator that maps a price vector onto the next "guess" of the price vector and show that this map has a fixed point using Kakutani's theorem.

Our innovation is to add as an equilibrium object mixed allocations over the optimal choice set, which act as tie-breaking criteria. This gives us a convex (probability) space of optimal choices and a convex set of macroeconomic variables. We show that this is sufficient to satisfy the conditions necessary for Kakutani's theorem to derive a stationary competitive equilibrium[5].

Definition. Given a price $\vec{p}$ and mixed allocation α, the household transition function for survivors, $GS_{\vec{p},\alpha} : \mathrm{S} \times \mathscr{B}(\mathrm{S}) \to [0,1]$ is defined as

$$GS_{\vec{p},\alpha}(s,S') = \int_{\Upsilon(s;\vec{p})} \int_{s' \in S'} 1\{h = h', j = j', a' = a+1\} \pi_I(i'|i) \pi_J(j'|j) ds' d\alpha(s,\vec{p})$$

The transition function for newborns, $GN : \mathscr{B}(\mathrm{S}) \to [0,1]$ is defined as[6]

$$GN(S') = b_1 \int_{s' \in S'} 1\{h' = 0, a' = 1\} \Pi_J(j') \Pi_I(i') ds'$$

The complete transition function, $G_{\vec{p},\alpha} : \mathrm{S} \times \mathscr{B}(\mathrm{S}) \to [0,1]$ is defined as

$$G_{\vec{p},\alpha}(s,S') = \lambda(a) GS_{\vec{p},\alpha}(s,S') + (1 - \lambda(a)) GN(S')$$

Given a price vector $\vec{p}$ and a mixed allocation α, the operator $\Upsilon_{\vec{p},\alpha} : \mathscr{M} \to \mathscr{M}$ is defined by the transition function G and gives the household distribution over the next period's states

$$\Upsilon_{\vec{p},\alpha}(\mu)(S') = \int_{s \in \mathrm{S}} G_{\vec{p},\alpha}(s,S') d\mu$$

Proposition 4. *[Existence of a unique invariant household distribution] For each $\vec{p} \in \mathrm{P}$ and $\alpha \in \Lambda$, there exists a unique $\mu_{\vec{p},\alpha} \in \mathscr{M}$ s.t. $\Upsilon_{\vec{p},\alpha}(\mu_{\vec{p},\alpha}) = \mu_{\vec{p},\alpha}$.*

Proof. We use Theorem 11.10 of Stokey et al. [21]. First, we show that $G_{\vec{p},\alpha}$ satisfies Doeblin's condition. From exercise 11.4g of Stokey et al. [21] (and using the fact that $GN(S')$ does not depend on the probability of survival $\lambda(a)$) , it is sufficient to show that GN satisfies Doeblin's condition. We must show that there exists a finite measure η on $(\mathrm{S}, \mathscr{B}(\mathrm{S}))$, an integer $N \geq 1$ and a number $n > 0$ such that if $\eta(S') \leq n$ then $GN^N(s,S') \leq 1 - n \; \forall s \in \mathrm{S}$. Set $\eta(S') = GN(S')$. Then we can see that GN satisfies Doeblin's condition for $N = 1$ and $n < 1/2$. This guarantees the existence of an invariant distribution.

Observe also that if $\eta(S') > 0$, then $G_{\vec{p},\alpha}(s,S') \geq (1 - \lambda(a)) GN(S') > 0$. This implies that the invariant distribution is unique. □

Lemma 5. *If $\{(s_n, \vec{p}_n)\}$ is a sequence in $\mathrm{S} \times \mathrm{P}$ converging to $(s_0, \vec{p}_0)$ then there exists a sequence $\{\alpha_n\}$ that converges to α_0 such that $G_{\vec{p}_n,\alpha_n}(s_n, \cdot)$ converges weakly to $G_{\vec{p}_0,\alpha_0}(s_0, \cdot)$.*

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[5] We will be dealing with convergent sequences in $\mathbb{R}^n$ throughout this proof. We follow the convention that if the space in question, $S \subseteq \mathbb{R}^n$, the metric is the standard metric on $\mathbb{R}^n$. In addition, if the space in consideration is a probability space then the corresponding metric is the sup-norm. Any non-standard metrics will be indicated in the proof.

[6] The initial location of the household is unimportant since it has no housing. However for completeness we need to choose some distribution, so we use Π_J, without loss of generality.

Proof. WLOG we can focus on sequences where the discrete states remain the same. Since Y is upper hemi-continuous in s and $\vec{p}$, $\exists(y_n(s_n,\vec{p}_n)) \to y(s_0,\vec{p}_0)$. Pick the indicator $\alpha(s_n,\vec{p}_n) = 1\{y_n(s_n,\vec{p}_n)\}$. Then,

$$\lim_{n\to\infty} GS_{\vec{p}_n,\alpha_n}(s_n,S') = GS_{\vec{p}_0,\alpha_0}(s_0,S') \quad \forall S' \in \mathscr{B}(\mathrm{S})$$

As GN is independent of $\vec{p}$ and s (it only depends on S') the result follows. □

Lemma 6. *Take a sequence of* $\{\vec{p}_n\} \in \mathrm{P} \to \vec{p}_0$. *Then,* $\exists\{\alpha_n\} \to \alpha_0$ *such that*

$$\mu_{\vec{p}_n,\alpha_n} \to \mu_{\vec{p}_0,\alpha_0}$$

where $\mu_{\vec{p}_n,\alpha_n} = \Upsilon_{\vec{p}_n,\alpha_n}\mu_{\vec{p}_n,\alpha_n}$ *and* $\mu_{\vec{p}_0,\alpha_0} = \Upsilon_{\vec{p}_0,\alpha_0}\mu_{\vec{p}_0,\alpha_0}$.

Proof. Proposition 4 and Lemma 5 are sufficient to use Theorem 12.13 of Stokey et al. [21] which gives us the result. □

So far we have shown that given any price vector, we can find a unique stationary distribution of households over the state space. Significantly, we have shown that the set of transition functions and the set of household distributions as a function of price are upper hemi-continuous. The next step is to define the aggregate variables and show that they are bounded as well. In what follows, we use $\mu_{\vec{p},\alpha}$ to represent the invariant household distribution given $\vec{p}$ and α.

Definition. Total housing demand on islands of quality j' is given by

$$H^d(j';\vec{p},\alpha) = \int_{\mathrm{S}}\int_{\Upsilon(s,\vec{p})} h(s,\vec{p})1\{j(s,\vec{p}) = j'\}d\alpha(s,\vec{p})d\mu_{\vec{p},\alpha}$$

Remark. H^d is continuous in α given $\vec{p}$.

Definition. The price transition function $\Omega^h_j : \mathrm{P}\times\Lambda \to \mathbb{R}_{++}$ is given by

$$\Omega^h_j = p(j) - \frac{H^d(j;\vec{p},\alpha) - \overline{H}\Pi_J(j)}{\overline{h}}p(j)$$

where $\overline{h}$ is the upper bound on housing choices.
$\Omega^h : \mathrm{P}\times\Lambda \to \mathrm{P}$ is defined as $\Omega^h(\vec{p},\alpha) = \{\Omega^h_j(\vec{p},\alpha)\}_{j\in J}$.
The *price transition correspondence*, $\Omega : \mathrm{P} \to \mathrm{P}$ is defined as

$$\Omega(\vec{p}) = \{\Omega^h(\vec{p},\alpha) : \alpha \in \Lambda\}$$

Define $\tilde{\Omega}^h_j(\vec{p}) = range_{\alpha\in\Lambda}\{\Omega^h_j(\vec{p},\alpha)\}$ and $\tilde{H}^d_j(\vec{p}) = range_{\alpha\in\Lambda}\{H^d(j,\vec{p},\alpha)\}$ $\forall j \in \mathrm{J}$.

Lemma 7. Ω*, is a self-map: range*$(\Omega) \subseteq \mathrm{P}$.

Proof. Let $\bar{l} = \max(l(i,j))$ and $\underline{l} = \min(l(i,j))$. Let $\hat{H}_j = \int_{\mathrm{S}} h\_ \cdot 1\{\hat{j} = j\_\}d\mu_{\vec{p},\alpha}$ so that $\hat{H}_j$ is the amount of housing on islands with quality j owned by surviving households entering the period.

Then equilibrium requires $\sum_{j\in J} p(j)(\bar{H}\Pi_J(j) - \hat{H}_j) < \bar{l}$. So there is some price, $\overline{p}$, at which all the housing on islands with quality j cannot be bought even if all households received the largest possible endowment and all other islands' prices were 0. Likewise, as the price $p(j) \to 0$, housing demand for age T households grows unbounded. Therefore, there is a lower bound on house prices $\underline{p} > 0$ beyond which housing demand is greater than $\bar{H}$. Henceforth then, $\mathrm{P} = [\underline{p}, \overline{p}]^J$. Then by construction, $range(\Omega_j^h) \subseteq [\underline{p}, \overline{p}]$. $\square$

Lemma 8. *Let $\Gamma : X \Rightarrow Y$, $f : X \times Y \to Z$, f continuous in y, and $\Gamma' : X \Rightarrow Z$, where $\Gamma'(x) = \{z : \exists y : y \in \Gamma(x)$ and $z = f(x,y)\}$. Then the following holds:*

(i) If Γ is compact-valued, then Γ' is compact-valued also.

(ii) If Γ is upper hemi-continuous, then Γ' is upper hemi-continuous also.

Proof. Using Ok [19], Prop. 3, Ch. D3, Pg 222, f takes compact sets to compact sets. Hence Γ' is compact-valued.

Pick $(x_m) \to x$ and $(z_m) \in \Gamma'(x_m) \forall m$. We want to show that there is a subsequence $(z_{m_k}) \to z \in \Gamma'(x)$. Since f is a function, for every z_m $\exists y_m : z_m = f(x_m, y_m)$, which implies that $y_m \in \Gamma(x_m)\ \forall m$. Since Γ is uhc, $\exists$ a subsequence $(y_{m_k}) \to y \in \Gamma(x)$. From continuity of f, the subsequence $z_{m_k} = f(x_{m_k}, y_{m_k}) \to z = f(x,y)$. So $z \in \Gamma'(x)$, hence Γ' is uhc. $\square$

Lemma 9. *$\tilde{H}_j^d$ are upper hemi-continuous, close-valued and convex valued.*

Proof. From Lemma 1, Y changes continuously in $\vec{p}$. Now let $\psi_p : \mathrm{P} \rightrightarrows \Delta$, where $\psi_p(\vec{p}) = \{\psi \in \Psi : supp(\psi) \subseteq \mathrm{Y}(s,\vec{p})\}$.

First, we show that ψ_p is upper hemi-continuous. Since ψ_p is closed, it is sufficient to show that ψ_p has a closed graph. Pick a sequence $\vec{p}_n \to \vec{p}$ and $\alpha_n \to \alpha_0$ with $\alpha_n \in \psi_p(\vec{p}_n)\ \forall n$. So, $supp(\alpha_n) \to supp(\alpha_0)$. Since Y is upper hemi-continuous and $\alpha_0 \in \psi_p$, $supp(\alpha_0(s,\vec{p})) \in \mathrm{Y}(s,\vec{p})$. Hence ψ_p is upper hemi-continuous.

From the definition of H^d, H^d is continuous. Hence, using Lemma 8, $\tilde{H}_j^d$ is upper hemi-continuous. Since ψ_p is close-valued and H^d is continuous, $\tilde{H}_j^d$ is close-valued. Since $H^d(j,\vec{p},\alpha)$ is linear in α, $\tilde{H}_j^d$ is convex-valued. $\square$

Lemma 10. *Ω is upper hemi-continuous, convex-valued and close-valued.*

Proof. H^d is continuous in α. Therefore Ω_j^h is continuous in α. Since Λ is a convex set, $\tilde{\Omega}_j^h$ is convex-valued. Ω_j^h is linear in H^d, therefore, Ω is convex-valued.

From the definition of Ω, it is a continuous transformation of Δ. Using Lemma 8, we get that Ω is upper hemi-continuous. Since Δ is compact and Ω is a continuous transformation, the image is compact as well. $\square$

Proposition 11. *A consolidated stationary equilibrium exists.*

Proof. P is a convex and compact space. Ω is convex-valued, and since it is upper hemi-continuous and compact-valued, it has a closed graph. Using Kakutani's FPT, $\exists \vec{p}^* \in \mathrm{P} :\ \vec{p}^* \in \Omega(\vec{p}^*)$. This implies that $\exists \alpha^* :\ \vec{p}^* = \Omega^h(\vec{p}^*, \alpha^*)$. $\square$

Finally we prove Lemma 3:

Proof. Take any CSE $\{\vec{p}^*, \tilde{Y}^*, Y^*, \alpha^*, \mu^*\}$. Let $\tilde{\mu}(s,\varepsilon,\mu_J) = \mu^*(s)\mu_J(j,\varepsilon)\Pi_J(j)$. Define the mixed allocation over $\tilde{Y}^*$ in the following way. For each $s \in S$, take the probability measure $\psi = \alpha^*(s,\vec{p}^*)$ and then for all $\tilde{y}^* \in \tilde{Y}^*$, set the value of the probability measure $\tilde{\alpha}^*((s,\varepsilon,\mu_J),\vec{p}^*)$ at $\tilde{y}^*$:

$$\tilde{\alpha}^*((s,\varepsilon\_,\mu_J),\vec{p}^*)(\tilde{y}^*) = \begin{cases} \psi(y^*(s,\vec{p}^*))1\{\varepsilon^*(\tilde{s},\vec{p}^*) = \varepsilon\} & \text{if } 1_m(s,y^*(s,\vec{p}^*)) = 0 \\ \psi(y^*(s,\vec{p}^*))\Pi_J(j^*(\tilde{s},\vec{p}^*))\mu_J(j^*(\tilde{s},\vec{p}^*),\varepsilon^*(\tilde{s},\vec{p}^*)) & \text{if } 1_m(s,y^*(s,\vec{p}^*)) = 1 \end{cases}$$

where y^* is the unique element of Y^* implied by $\tilde{y}^*$ and 1_m is a moving indicator. Then all markets clear. □

Acknowledgement. I thank John Leahy, Morten Ravn, Tom Sargent and Ennio Stacchetti for helpful comments. This paper is adapted from work with Gopal Vasudev cited herein. The author thanks the ESRC for funding under grant number PTA-026-27-2395.

[1] S.R. Aiyagari. Uninsured idiosyncratic risk and aggregate saving. *Quarterly Journal of Economics*, 109:659–684, 1994.

[2] Fernando Alvarez and Marcelo Veracierto. Fixed-term employment contracts in an equilibrium search model. *Journal of Economic Theory*, 147(5):1725 – 1753, 2012.

[3] Eduardo M. Azevedo, E. Glen Weyl, and Alexander White. Walrasian equilibrium in large, quasi-linear markets. *Theoretical Economics*, Forthcoming.

[4] Erik J. Balder. More on equilibria in competitive markets with externalities and a continuum of agents. *Journal of Mathematical Economics*, 44(7-8):575–602, July 2008.

[5] Truman F. Bewley. Notes on stationary equilibrium with a continuum of independently fluctuating consumers. Yale University, 1984.

[6] Matthew Chambers, Carlos Garriga, and Don E. Schlagenhauf. Accounting for changes in the homeownership rate. *International Economic Review*, 50(3):677–726, 08 2009.

[7] Yongsung Chang and Sun-Bin Kim. From individual to aggregate labor supply: A quantitative analysis based on a heterogeneous agent macroeconomy. *International Economic Review*, 47(1):1–27, 02 2006.

[8] Satyajit Chatterjee, Dean Corbae, Makoto Nakajima, and Jose Victor Rios-Rull. A quantitative theory of unsecured consumer credit with risk of default. *Econometrica*, 75:1525–1589, 2007.

[9] Jonathan Halket and Santhanagopalan Vasudev. Saving Up or Settling Down: Home Ownership over the Life Cycle. *Review of Economic Dynamics*, 17(2):345–366, April 2014.

[10] Mark Huggett. The risk-free rate in heterogeneous-agent incomplete-insurance economies. *Journal of Economic Dynamics and Control*, 17(5-6):953–969, 1993.

[11] Ayse Imrohoruglu. Cost of business cycles with indivisibilities and liquidity constraints. *Journal of Political Economy*, 97(6):1364–83, December 1989.

[12] Kenneth L. Judd. The law of large numbers with a continuum of iid random variables. *Journal of Economic Theory*, 35(1):19–25, February 1985.

[13] M. Ali Khan. On cournot-nash equilibrium distributions for games with a nonmetrizable action space and upper semicontinuous payoffs. *Transactions of the American Mathematical Society*, 315(1):pp. 127–146, 1989.

[14] Sagiri Kitao. Entrepreneurship, taxation and capital investment. *Review of Economic Dynamics*, 11(1):44–69, January 2008.

[15] Per Krusell and Anthony A. Smith Jr. Income and wealth heterogeneity in the macroeconomy. *Journal of Political Economy*, 106(5):867–896, October 1998.

[16] Robert E. Lucas and Edward Prescott. Equilibrium search and unemployment. *Journal of Economic Theory*, 7(2):188–209, 1974.

[17] Andreu Mas-Colell. Indivisible commodities and general equilibrium theory. *Journal of Economic Theory*, 16(2):443–456, December 1977.

[18] Andreu Mas-Colell. On a theorem of schmeidler. *Journal of Mathematical Economics*, 13 (3):201 – 206, 1984.

[19] Efe A. Ok. *Real Analysis with Economic Applications*. Number mathecon1 in Online economics textbooks. SUNY-Oswego, Department of Economics, June 2004.

[20] Kali P. Rath. Existence and upper hemicontinuity of equilibrium distributions of anonymous games with discontinuous payoffs. *Journal of Mathematical Economics*, 26(3):305–324, 1996.

[21] Nancy Stokey, Robert Lucas, and Edward Prescott. *Recursive Methods In Economic Dynamics*. Havard University Press, 1989.

[22] Yeneng Sun. The exact law of large numbers via fubini extension and characterization of insurable risks. *Journal of Economic Theory*, 126(1):31 – 69, 2006.

[23] William R. Zame and Mitsunori Noguchi. Competitive markets with externalities. *Theoretical Economics*, 1(2):143–166, June 2006.