Appendix (FOR ONLINE PUBLICATION)

Supplemental Data Analysis

The correlation between rent-to-price ratios and homeownership rates

In the body of the paper, we report the "stylized" fact that the rent-to-price ratio is negatively correlated with ownership rates. Renters tend to rent in submarkets where renting is relatively expensive as compared to owning. Here we check if this fact may be driven by some underlying correlations that would be a challenge to fit within the model. Specifically, we check whether this fact is present in alternative data sets and also across some obvious geographic partitions of markets.

To corroborate our Craigslist data, we use data from Zillow[29] on price to rent ratios at the zip code level. Zillow computes the ratio differently from us: "The ratio is calculated at the house level first, where the estimated home value is divided by 12 times the estimated rent price. Then the median of all house level price to rent ratios for a given region [zip code] is calculated." As this quote says, Zillow forms a price to rent ratio by comparing prices and rents of similar houses. However, how it decides which houses on either market are comparable to each other is proprietary and thus a black box to us. Nevertheless, it means that some difference in the distribution of houses across tenure but within zip code is controlled for.

Table 4 lists the correlations between the log rent-to-price ratio and the homeownership rate within MSAs at the zip code level for the Zillow data and at the zip x bedroom level for the Craigslist sample. We list the top MSAs according to unique rental vacancies in the Craigslist data over the sample (according to the algorithm described in the paper). In order to see whether the correlation between rent-to-price ratios and homeownership is different within central urban areas (as opposed to suburbs), the table also lists the correlation for those same MSAs using only those zip codes within the central county or city for a given MSA.

[29]http://www.zillow.com/research/data/

Table 4: Correlation between rent-to-price ratios and homeownership: Within Largest MSAs, Counties and Cities

	MSA		County		City	
MSA	Zillow	Craigslist	Zillow	Craigslist	Zillow	Craigslist
Atlanta-Sandy Springs-Marietta	-0.19	-0.15	-0.51	-0.41	-0.55	-0.42
Chicago-Naperville-Joliet	-0.18	-0.05	-0.10	-0.09	-0.01	-0.19
Denver-Aurora	-0.21	-0.26	0.00	-0.31	0.00	-0.31
Los Angeles-Long Beach-Santa Ana	-0.17	-0.19	-0.11	-0.13	-0.06	-0.15
Minneapolis-St. Paul-Bloomington	-0.09	-0.02	-0.17	0.04	-0.03	0.17
Philadelphia-Camden-Wilmington	-0.46	-0.10	0.04	0.11	0.04	0.11
Phoenix-Mesa-Scottsdale	-0.39	-0.44	-0.44	-0.46	-0.06	-0.46
Riverside-San Bernardino-Ontario	-0.24	-0.38	NA	NA	NA	NA
San Francisco-Oakland-Fremont	-0.08	-0.30	-0.19	-0.21	-0.19	-0.21
Seattle-Tacoma-Bellevue	-0.10	-0.14	-0.09	-0.25	-0.19	-0.05
Tampa-St. Petersburg-Clearwater	-0.42	-0.27	-0.53	-0.43	-0.50	-0.15
Washington-Arlington-Alexandria	-0.17	-0.19	-0.42	-0.40	-0.42	-0.40

Riverside-San Bernardino-Ontario, CA MSA has no city in the MSA with a large number of zip codes. For the other MSAs, the county (city) for each MSA are, in table order: Fulton (Atlanta), Cook (Chicago), Denver (Denver), Los Angeles (Los Angeles), Hennepin (Minneapolis), Philadelphia (Philadelphia), Maricopa (Phoenix), San Francisco (San Francisco), King (Seattle), Hillsborough (Tampa), District of Columbia (Washington D.C.). The city and county are the same geography for Denver, Philadelphia, San Francisco and Washington D.C..

The correlations within MSAs in Table 4 are uniformaly negative in both the Craigslist and Zillow data. Moreover, for most MSAs, the estimated correlations are numerically similar. Using only the main city or county in each MSA, some correlations become less negative than their MSA-level counterparts and in two out of 11 cities the correlation is slightly positive. However in many cases, the correlation becomes more negative. The number of zip codes in the city is of course quite a bit smaller than the total number in the whole MSA, so it is not surprising that there is a wider variance of estimated correlations at the city level. In summary, the pattern of relative rents and prices in Craigslist is broadly similar to the same patter in the Zillow data and in both data sets the same correlation between homeownership and rent-to-price ratios is evident over the same subsamples.

Regression results using an alternative subsample

In this section, we repeat the same regressions as in the main text of the paper using only a subsample of the data. Here we restrict the subsample to only those MSAs that have at least 1000 rental and for-sale listings in our sample to ensure that our main results are not being driven by random pathological markets. The list of MSA is as in Table 4. Comparing Tables 5 and 6 to their counterparts in the main text, it is evident that the results are robust to restricting the sample. Most coefficients are statistically identical to those in the main text; only the effect of *duration in home* on rents is marginally different.

Table 5: Regressions of prices on vacancy durations and durations in home: alternative subsample

	rent	rent	rent	price	price	price
time rent	0.031*					
	(0.013)					
time sale				0.099*		
				(0.039)		
prop over 35 yr		-0.214**			0.441	
		(0.034)			(0.320)	
duration in home			-0.008			0.142*
			(0.015)			(0.064)
median inc	0.374**	0.380**	0.372**	0.754**	0.737**	0.800**
	(0.081)	(0.018)	(0.020)	(0.036)	(0.036)	(0.049)
R^2	0.88	0.88	0.88	0.78	0.79	0.78
N	9202	9202	8846	8217	8217	7886

All variables are in logs and all regressions include dummies for each market. *time rent* (*time sale*) is the T_i conditional on being for-rent (for-sale). *rent* (*price*) is the final listed rent (price) for the property. *prop over 35 yr* is the proportion of households in the zip code with head of house age 35 or over. SEs, clustered by market, in parentheses. ** $p < 0.01$, * $p < 0.05$.

Table 6: Regressions of vacancy duration and homeownership rates on durations in home: alternative subsample

	time rent	time rent	time sale	time sale	homeownership	homeownership
prop over 35 yr	-0.067**		-0.074		0.671**	
	(0.024)		(0.085)		(0.110)	
duration in home		-0.066**		0.003		0.289**
		(0.014)		(0.018)		(0.047)
median inc	-0.008	-0.020	-0.003	0.754**	0.279**	0.368**
	(0.010)	(0.011)	(0.013)	(0.036)	(0.032)	(0.023)
R^2	0.35	0.35	0.35	0.36	0.81	0.81
N	9202	8846	8218	7887	9617	9264

All variables are in logs and all regressions include dummies for each market. *time rent* (*time sale*) is the T_i conditional on being for-rent (for-sale). *homeownership* is the homeownership rate in the cell. *prop over 35 yr* is the proportion of households in the zip code with head of house age 35 or over. SEs, clustered by market, in parentheses. ** $p < 0.01$, * $p < 0.05$.

Definition of Competitive Equilibrium With Renting and Owning

Definition 4. *A competitive search equilibrium with renting, owning and private information is a set* $\{Z^{*i}_{po}, Z^i_o, \tilde{Z}^i_p\}_{i\in\mathbb{I}}$ *with* $Z^{*i}_{po}, Z^i_o : [\underline{\chi}, \overline{\chi}] \to \mathbb{R}_{++}$ *and* $\tilde{Z}^i_p \in \mathbb{R}_{++}$*, a set of incentive compatible rents* $\tilde{\mathrm{W}}^*_{\mathrm{p}} \subseteq [\rho H, h]^{\mathbb{I}}$*, a set of prices* $P^* = \{P^{*i}\}_{i\in\mathbb{I}} \in [H, h/\rho]^{\mathbb{I}}$*, a measure* λ_r *on* $[\rho H, h]$ *with support* $\tilde{\mathrm{W}}^*_{\mathrm{p}}$*, a measure* λ_o *on* $[H, h/\rho]$ *with support* P^**, functions* $\tilde{\theta}^*_p : [\rho H, h] \to \mathbb{R}_+$ *and* $\theta^*_o : [H, h/\rho] \to \mathbb{R}_+$ *and functions* $\psi_r : [\rho H, h] \to \Delta^{I\times[\underline{\chi},\overline{\chi}]}$ *and* $\psi_o : [H, h/\rho] \to \Delta^{I\times[\underline{\chi},\overline{\chi}]}$ *satisfying:*

(i) Landlords' profit maximization and free entry: for any $w \in [\rho H, h]$

$$\left[1 + \left(\alpha_l(\tilde{\theta}^*_p(w)) \sum_{i\in\mathbb{I}} \frac{\int_{[\underline{\chi},\overline{\chi}]} \psi_{r,(i,\chi)}(w) dF(\chi)}{\rho + \gamma_i}\right)^{-1}\right]^{-1} w \leq \rho H$$

with equality if $w \in \tilde{\mathrm{W}}^*_{\mathrm{p}}$.

(ii) Builders' profit maximization and free entry: for any $P \in [H, h/\rho]$

$$\frac{\alpha_l(\theta^*_o(P))}{\rho + \alpha_l(\theta^*_o(P))} P \leq H$$

with equality if $P \in P^*$.

(iii) Households' optimal search: Let

$$\tilde{Z}^i_p \equiv \max_{w'\in\tilde{\mathrm{W}}^*_{\mathrm{p}}} \frac{1}{\rho} \frac{\alpha_l(\tilde{\theta}^*_p(w'))}{\tilde{\theta}^*_p(w')(\rho+\gamma_i) + \alpha_l(\tilde{\theta}^*_p(w'))}(h - w')$$

$$Z^i_o \equiv \max_{P'\in P^*} \frac{1}{\rho}\left(1 + \frac{\rho+\gamma_i}{\alpha_l(\theta^*_o(P'))/\theta^*_o(P')}\right)^{-1}\left[h - \left(1 + \frac{\gamma_i}{\rho + \alpha_l(\theta^*_o(P'))}\right)\rho P'\right]$$

$$\text{and} \quad Z^{*i}_{po}(\chi) = \max\{Z^i_o - \chi, \tilde{Z}^i_p\} \ \forall \ i \in \mathbb{I}$$

Then $\forall w \in [\rho H, h]$ *and* $\forall \gamma_i$

$$Z^{*i}_{po}(\chi) \geq \frac{1}{\rho} \frac{\alpha_l(\tilde{\theta}^*_p(w))}{\tilde{\theta}^*_p(w)(\rho+\gamma_i) + \alpha_l(\tilde{\theta}^*_p(w))}(h - w)$$

with equality if $\tilde{\theta}^*_p(w) > 0$ *and* $\psi_{r,(i,\chi)}(w) > 0$*. And* $\forall P \in [H, h/\rho]$ *and* $\forall \gamma_i$

$$Z^{*i}_{po}(\chi) \geq \frac{1}{\rho}\left(1 + \frac{\rho+\gamma_i}{\alpha_l(\theta^*_o(P))/\theta^*_o(P)}\right)^{-1}\left[h - \left(1 + \frac{\gamma_i}{\rho + \alpha_l(\theta^*_o(P))}\right)\rho P\right] - \chi$$

with equality for $\chi = \underline{\chi}$ *if* $\theta^*_o(P) > 0$ *and* $\psi_{o,(i,\chi)}(P) > 0$.

(iv) market clearing:

$$\int_{\tilde{\mathrm{W}}^*_{\mathrm{p}}} \int_{[\underline{\chi},\overline{\chi}]} \psi_{r,(i,\chi)}(w)\tilde{\theta}^*_p(w)dF(\chi)d\lambda_r(w) + \int_{P^*} \int_{[\underline{\chi},\overline{\chi}]} \psi_{o,(i,\chi)}(P)\theta^*_o(P)dF(\chi)d\lambda_o(P) = \pi_i \quad \forall i$$

Proofs not in the main text

Proof of Lemma 1

Let w be any contract in any pooling equilibrium for which there exists $i \neq j$ and $\psi_i(w) > 0$, $\psi_j(w) > 0$. The landlord takes the expected values $\rho Z_r(\gamma_i, r_i, \theta(\mathrm{w}))$ and $\rho Z_r(\gamma_j, r_j, \theta(\mathrm{w}))$ of the two types as given.

A landlord cannot make strictly lower expected profits from either type. If she could, then a deviating contract would be the menu that does not offer an attractive rent to that type. By rational expectations, the expected queue length must be the same and so the landlord will make strictly higher expected profits, a contradiction. Therefore:

$$\frac{\alpha_l(\theta(\mathrm{w}))}{\rho + \gamma_i + \alpha_l(\theta(\mathrm{w}))} r_i = \frac{\alpha_l(\theta(\mathrm{w}))}{\rho + \gamma_j + \alpha_l(\theta(\mathrm{w}))} r_j = \rho H \tag{13}$$

The lemma follows trivially from there. □

Proof of Proposition 2

We want to prove the existence and uniqueness of the solution of the unconstrained maximization problem. We follow the following steps (and drop dependence on i)

The landlord's zero profit constraint (ZPC) constraint holds with equality for each type: Suppose not. We can increase Z by decreasing w and/or θ in a ball $B_\varepsilon(w^*_r, \theta^*_r)$ and still meet the constraint for ε small enough. Thus (w^*_r, θ^*_r) is not a maximum.

Existence. We can impose the ZPC with equality: $\theta_r^{zpc}(\gamma, w) = \alpha^{-1}\left(\frac{(\rho+\gamma)\rho H}{w-\rho H}\right)$. The maximization problem simplifies to: $\max_{w\in[\rho H,h]} Z_r^{zpc}(\gamma, w) = Z_r(\gamma, w, \theta_r^{zpc}(\gamma, w))$. Note that as $w \to \rho H$, $\theta_r^{zpc}(\gamma, w) \to \infty$ and $\frac{\alpha(\theta_r^{zpc})}{\theta_r^{zpc}}(\gamma, w) \to 0$, thus $Z_r^{zpc}(\gamma, w = \rho H) = 0$. The objective function is continuous and the constraint set is compact.

The solution is interior. From above, $Z_r^{zpc}(\gamma, w = \rho H) = 0$ and it is easy to show that $Z_r^{zpc}(\gamma, w = h) = 0$. Moreover, $Z_r^{zpc}(\gamma, w) > 0$ for all $w \in (\rho H, h)$.

Uniqueness. Analytically, it is easier to solve the equivalent problem:

$$\max_{\theta\in\mathbb{R}_+} Z_r(\gamma, w_r^{zpc}(\gamma, \theta), \theta)$$

where w_r^{zpc} satisfies the ZPC. The objective function is non-negative iff $\alpha \geq \frac{(\rho+\gamma)\rho H}{h-\rho H}$, or equivalently $\theta \geq \alpha^{-1}\left(\frac{(\rho+\gamma)\rho H}{h-\rho H}\right)$, and $\lim_{\theta\to\infty} Z_r(\gamma, w_r^{zpc}(\gamma, \theta), \theta) = 0$. Since the objective function is continuously differentiable on $\mathbb{R}_+$, the first-order condition is necessary for an optimum:

$$\frac{h}{\rho H} = 1 + \frac{1}{\theta_r^*}\frac{\varepsilon}{1-\varepsilon} + \frac{\rho+\gamma}{\alpha_l(\theta_r^*)(1-\varepsilon)} \tag{14}$$

The right-hand side of (14) is a decreasing, continuous, function in θ. Thus, there is only one solution θ^* of the maximization problem. □

Proof of Result 1

From (14), θ_r^* is increasing in γ, so from the zero-profit condition for landlords, w_r^* is increasing in γ. So Z_r^* is decreasing in γ. □

Proof of Proposition 3

We go through the following steps:

The $IC(j,i)$ with $j > i$, never binds; a type with $\gamma_j < \gamma_i$ never wants to deviate to the i-contract. Any contract and associated market-tightness for a type i is also feasible for any type $j > i$.

For all $\{PR_i\}$, the ZPC binds and, for $i > 1$, at least one IC must bind.

By contradiction. Suppose not. If no constraint ever binds, then Z_p^{*i} is maximized by setting $w = \theta = 0$, but that violates the ZPC. If only the ZPC binds, then the problem is equivalent to the unconstrained one, but in that case the optimal contract associated with higher i (lower γ_i) is always preferred by all $j < i$, thus the IC is violated. If one $IC(j,i)$ binds but not the ZPC, then by the sorting condition we can pick a couple $(w,\theta) \in B_\varepsilon((w_p^{*i}, \theta_p^{*i}))$ such that the ZPC still holds and both types i and j are strictly better off, thus that is not a solution.

$\{PR_1\}$ is equivalent to the first best problem

Follows from the previous results.

There exists an unique solution to $\{PR_i\}$ for all $i > 1$. At the optimum, only the marginal IC is binding, $IC(i-1,i)$.

We prove this iteratively.

First step. The solution for $i = 1$ is the first best allocation: $Z_p^{*1} = Z_r^{*1}$, $\theta_p^{*1} = \theta_r^{*1}$ and $w_p^{*1} = w_r^{*1}$.

Iterative step. Consider the problem PR_i for type $i > 1$. We go through two sub-steps.

i Assume first that only the marginal IC is binding, $IC(i-1,i)$. By the previous analysis, this must be the case, in particular, for $i = 2$. The constrained optimum Z_p^{*i}, market tightness θ_p^{*i} and rent w_p^{*i} must satisfy the ZPC and $IC(i-1,i)$. Thus, θ_p^{*i} and w_p^{*i} satisfy the following non-linear system in θ and w:

$$X(\gamma_i, w, \theta) = H$$
$$Z_r(\gamma_{i-1}, w, \theta) = Z_p^{*(i-1)}$$

We can express w as a function of θ in both equations:

$$w = w_{zpc}(\gamma_i, \theta) = \left(1 + \frac{\rho + \gamma_i}{\alpha}\right)\rho H \tag{15}$$
$$w = w_{icc}(\gamma_{i-1}, \theta) = h - \left(1 + \frac{\rho + \gamma_{i-1}}{\alpha/\theta}\right)\rho Z_p^{*(i-1)} \tag{16}$$

Equation (16) is the indifference curve of type $(i-1)$ that by construction goes through the optimal point $(\theta_p^{*(i-1)}, w_p^{*(i-1)})$. Moreover, at $(\theta_p^{*(i-1)}, w_p^{*(i-1)})$ landlords make zero profits in the market for type $(i-1)$, thus they make strictly positive profits with households of type i. It implies that, at $\theta_p^{*(i-1)}$, the zero profit curve in the market for type i (15) is met for a lower value of the rent, $w < w_p^{*(i-1)}$. Thus:

$$w_{zpc}(\gamma_i, \theta_p^{*(i-1)}) < w_{icc}(\gamma_{i-1}, \theta_p^{*(i-1)})$$

At the limit, $w_{zpc} > w_{icc}$:

$$\lim_{\theta^{zp} \to 0} w^{zp} = \infty > h - \rho Z_r^{*1} = \lim_{\theta^{ic} \to 0} w^{ic}$$
$$\lim_{\theta^{zp} \to \infty} w^{zp} = \rho H > -\infty = \lim_{\theta^{ic} \to \infty} w^{ic}$$

Thus, they cross at least twice, one time on the left and one time on the right of the point $(\theta_p^{*(i-1)}, w_p^{*(i-1)})$.

It is easy to show that:

Result 4. *The expected value of a type i increases as θ increases on the indifference curve of a type j, with $i > j$ ($\gamma_i < \gamma_j$), and viceversa; moreover, the two types have the same expected values at $\theta = 0$.*

Intuitively, a higher market tightness affects more the type with higher moving probability. This implies that the expected value of type i is maximized at the crossing point with higher θ and lower w, and it is higher than the optimal expected value of type $(i-1)$:

$$\theta_p^{*i} > \theta_p^{*(i-1)}$$
$$w_p^{*i} < w_p^{*(i-1)}$$
$$Z_p^{*i} > Z_p^{*(i-1)}$$

This solves the problem for $i = 2$.

(ii) In general, we need to show that no other $IC(i-k, i)$ binds, with $i > 2$ and $k > 1$. Suppose by way of contradiction that it does bind. We can assume, from substep (i), that (only) the marginal incentive compatibility constraints bind for all $j < i$, in particular $IC(i-k, i-k+1)$. Thus, type $(i-k)$ is indifferent between the pairs $(\theta_p^{*(i-k)}, w_p^{*(i-k)})$, $(\theta_p^{*(i-k+1)}, w_p^{*(i-k+1)})$ and $(\theta_p^{*i}, w_p^{*i})$. Since the pair $(\theta_p^{*(i-k+1)}, w_p^{*(i-k+1)})$ is feasible for type i (the zero profit condition for type i is not binding), by result 4 type i chooses optimally a higher θ and lower w:

$$\theta_p^{*i} > \theta_p^{*(i-k+1)} > \theta_p^{*(i-k)}$$
$$w_p^{*i} < w_p^{*(i-k+1)} < w_p^{*(i-k)}$$

But then, by the same argument, type $(i-k+1)$ would prefer $(\theta_p^{*i}, w_p^{*i})$ to $(\theta_p^{*(i-k+1)}, w_p^{*(i-k+1)})$, violating the incentive compatibility constraint $IC(i-k+1, i)$. Thus $(\theta_p^{*i}, w_p^{*i})$ is not incentive compatible. A contradiction. □

Proof of Proposition 4

The proof is divided into two main parts. Part (1) proves that, if an allocation solves (PR), then there exists a competitive search equilibrium with that allocation. Part (2) proves that any equilibrium allocation solves (PR). From Proposition 3, it follows that the equilibrium exists and is unique.

Part (1)

The proof is by construction. Let $\{w_p^{*i}, \theta_p^{*i}\}_{\mathbb{I}}$ be a solution to the (PR) problem. Construct the candidate equilibrium allocation as follows:

$$Z_p^{*i} = Z_r(\gamma_i, w_p^{*i}, \theta_p^{*i}) \quad \forall i$$
$$W_p^* = \{w_p^{*i}\}_{\mathbb{I}}$$

Let the functions θ_p^* and Ψ be defined over the entire set $[\rho H, h]$ as follows:

$$\theta_p^*(w): \quad \frac{\alpha(\theta_p^*(w))}{\theta_p^*(w)} = \min_{j\in\mathbb{I}} \left[\frac{h-w}{\rho Z_p^{*j}} - 1\right]^{-1}(\rho+\gamma_j)$$
$$\psi_k(w) = 1 \quad \text{implies} \quad k = \arg\min_{j\in\mathbb{I}} \left[\frac{h-w}{\rho Z_p^{*j}} - 1\right]^{-1}(\rho+\gamma_j)$$

If there is more than one solution k that minimizes that equation, choose the largest one. The definition of the function $\Psi(w)$ then implies $\psi_j(w_i^*) = 0$ for all $j \neq k$. The expression for ρZ_p^{*i} implies:

$$\theta_p^*(w_p^{*i}) = \theta_p^{*i} \quad \forall w_p^{*i} \in W_p^*$$
$$\psi_i(w_p^{*i}) = 1 \quad \forall w_p^{*i} \in W_p^*$$

The first equation is derived by noting that if the expression is minimized for $j \neq i$, then j strictly prefers the i-optimal contract to the j-optimal contract, a contradiction. The second equation follows, noting that, by the properties of the constrained optimum, the equation is minimized by i and $(i-1)$ only. Finally, the measure of landlords posting w_p^{*i} is consistent with market tightness $\Theta(w_p^{*i})$:

$$\lambda(w_p^{*i}) = \frac{\psi_i}{\theta_p^*(w_p^{*i}) + \frac{\alpha(\theta_p^*(w_p^{*i}))}{\gamma_i}} \quad \forall w_p^{*i} \in W_p^*$$

and $\lambda(w) = 0$ if $w \notin W_p^*$.

We prove that this allocation satisfies all the equilibrium conditions:

(i) Landlords' profit maximization and free entry.

By construction, the ZPC holds with equality $\forall w \in W_p^*$. Consider $w \notin W_p^*$, $w \in [\rho H, h]$ and assume, by contradiction:

$$\left[1 + \left(\alpha_l(\theta_p^*(w)) \sum_{i\in\mathbb{I}} \frac{\psi_i(w)}{\rho+\gamma_i}\right)^{-1}\right]^{-1} w > \rho H$$

This implies $\theta_p^*(w) > 0$ and there exists j with $\psi_j(w) > 0$ and

$$\left[1 + \frac{\rho + \gamma_j}{\alpha_l(\theta_p^*(w))}\right]^{-1} w > \rho H$$

By construction of $\Psi(w)$, $\psi_j(w) = 1$ and $\psi_k(w) = 0$ $\forall k \neq j$. Then, by construction of $\Theta(w)$:

$$\frac{\alpha(\theta_p^*(w))}{\theta_p^*(w)} = \left[\frac{h - w}{\rho Z_p^{*j}} - 1\right]^{-1} (\rho + \gamma_j) \leq \left[\frac{h - w}{\rho Z_p^{*k}} - 1\right]^{-1} (\rho + \gamma_k) \quad \forall k$$

And the inequality holds strictly for all $k > j$.

So, the couple $(w, \theta_p^*(w))$ satisfies all the constraints of the problem (P_j) and guarantees the optimal value Z_p^{*j} to j and strictly positive profits to landlords. By continuity and the sorting condition, there exists a couple $(w', \theta') \in B_\varepsilon(w, \theta_p^*(w))$, with $w' < w$ and $\theta' > \theta_p^*(w)$ such that $Z_r(\gamma_j, w', \theta') > Z_p^{*j}$ and the ZPC and IC's are satisfied. A contradiction.

(ii) Households' optimal search.

By construction, $Z_p^{*i} = \max_{w \in W_p^*} Z_r(\gamma_i, w, \theta_p^*(w))$, $\theta_p^*(w_p^{*i}) > 0$ and $\psi_i(w_p^{*i}) > 0$. Moreover, by the construction of $\theta_p^*(w)$, for all $w \in [\rho H, h]$, $Z_p^{*i} \geq \frac{1}{\rho} \frac{\alpha_l(\theta_p^*(w))}{\theta_p^*(w)(\rho+\gamma_i)+\alpha_l(\theta_p^*(w))}(h - w)$.

(iii) Market clearing.

Follows directly by construction.

Part (2)

Part (i) of the equilibrium definition implies that $\theta_p^*(w) > 0$ for all $w \in W_p^*$, and part (iii) implies that for each i $\exists$ $w \in W_p^*$ such that $\psi_i(w) > 0$. It follows that, $\forall i$, $\exists$ $w \in W_p^*$ such that $\theta_p^*(w) > 0$ and $\psi_i(w) > 0$, thus from condition (ii) $Z_r(\gamma_i, w, \theta_p^*(w)) = Z_p^{*i}$.

We go through four steps to show that the equilibrium allocation solves the constrained maximization problem P_i, for all i:

(i) The ZPC is satisfied.

Let $w_p^{*i} \in W_p^*$ and $\theta_p^{*i} \equiv \theta_p^*(w_p^{*i})$, with $\psi_i(w_p^{*i}) > 0$. Suppose by contradiction that the ZPC is not satisfied:

$$\left[1+\frac{\rho+\gamma_i}{\alpha_l(\theta_p^{*i})}\right]^{-1} w_p^{*i} < \rho H$$

Then, by equilibrium condition (i) and by noting that expected profits are decreasing in γ, there exists a $k > i$ such that:

$$\left[1+\frac{\rho+\gamma_k}{\alpha_l(\theta_p^{*i})}\right]^{-1} w_p^{*i} < \rho H$$

By the sorting condition, $\exists\,(\theta', w') \in B_\varepsilon$, with $\theta' > \theta$ and $w' < w$ s.th.:

$$Z_r(\gamma_j, w', \theta') > Z_r(\gamma_j, w_p^{*i}, \theta_p^{*i}) \quad \forall j \geq k$$
$$Z_r(\gamma_j, w', \theta') < Z_r(\gamma_j, w_p^{*i}, \theta_p^{*i}) \quad \forall j < k$$

Thus, for all $j < k$, $Z_r(\gamma_j, w', \theta') < Z_r(\gamma_j, w_p^{*i}, \theta_p^{*i}) \leq Z_p^{*j}$ by equilibrium condition (ii). But then condition (ii) and $\theta' > 0$ imply $\psi_j(w') = 0$, $\forall\, j < k$. It follows:

$$\left[1+\left(\alpha_l(\theta')\sum_{i\in\mathbb{I}}\frac{\psi_i(w')}{\rho+\gamma_i}\right)^{-1}\right]^{-1} w' \geq \left[1+\frac{\rho+\gamma_h}{\alpha_l(\theta')}\right]^{-1} w' > \rho H$$

where the last inequality holds for ε small enough. Thus, (w', θ') is a profitable deviation for the landlord. A contradiction.

(ii) IC's are satisfied.

Consider again $w_p^{*i} \in W_p^*$, $\theta_p^{*i} \equiv \theta_p^*(w_p^{*i}) > 0$ and $\psi_i(w_p^{*i}) > 0$. By equilibrium condition (ii), applied to all types j, it must be that:

$$Z_r(\gamma_j, w_p^{*i}, \theta_p^{*i}) \leq Z_p^{*j} \quad \forall j$$

Thus, the incentive compatibility constraints $IC(j, i)$ are satisfied $\forall j$.

(iii) The equilibrium value is equal to Z_p^{*i}, as defined in equilibrium condition (ii).
Again, it follows directly from condition (ii), since $\theta_p^*(w_p^{*i}) > 0$ and $\psi_i(w_p^{*i}) > 0$.

(iv) The equilibrium allocation solves P_i.

Let $\bar{Z}_r^i$ be the value from the competitive equilibrium allocation for each i. Suppose there exists a (w, θ) which respects the constraints for PR_i and is better: $X_r(w, \theta) \geq H$, $Z_r(\gamma_i, w, \theta) > \bar{Z}_r^i$ and $Z_r(\gamma_j, w, \theta) \leq \bar{Z}_r^j$ for $j < i$.

Take $w' \in B_\epsilon(w)$ such that $X_r(w', \theta) > X_r(w, \theta)$, $Z_r(\gamma_i, w', \theta) > \bar{Z}_r^i$ and $Z_r(\gamma_j, w', \theta) \leq \bar{Z}_r^j$ for $j < i$. There exists a $B_{\epsilon'}(w', \theta)$ such that for all $(\hat{w}, \hat{\theta}) \in B_{\epsilon'}(w', \theta)$, $X_r(\hat{w}, \hat{\theta}) > X_r(w, \theta)$ and $Z_r(\gamma_i, \hat{w}, \hat{\theta}) > \bar{Z}_r^i$.

By sorting (relative to (w', θ)), there exists $(w'', \tilde{\theta}) \in B_{\epsilon'}(w', \theta)$ such that $Z_r(\gamma_i, w'', \tilde{\theta}) > \bar{Z}_r^i$ and $Z_r(\gamma_j, w'', \tilde{\theta}) < \bar{Z}_r^j$ for $j < i$. Note that $w'' < w'$ and $\tilde{\theta} > \theta$.

The equilibrium θ for the rent w'' according to the competitive equilibrium: $\theta_p^*(w'') \geq \tilde{\theta}$. So $Z_r(\gamma_j, w'', \theta_p^*(w'')) < \bar{Z}_r^j$ for $j < i$ and $X_r(w'', \theta_p^*(w'')) \geq X_r(w'', \tilde{\theta}) \geq X_r(w', \theta) > H$. So the allocation which gave $\bar{Z}_r^i$ was not an equilibrium allocation. □

Proof of Result 2

Start from the two equations for the constrained optimum and write them in Δ-form:

$$w(\gamma_{i+1} - \Delta) = \left(1 + \frac{\rho + \gamma_{i+1} - \Delta}{\alpha(\gamma_{i+1} - \Delta)}\right)\rho H$$
$$w(\gamma_{i+1} - \Delta) = h - \left(1 + \frac{\rho + \gamma_{i+1}}{\alpha(\gamma_{i+1} - \Delta)/\theta(\gamma_{i+1} - \Delta)}\right)\rho Z_p^{*(i+1)}$$

where $\alpha(\gamma_{i+1} - \Delta) = \alpha(\theta(\gamma_{i+1} - \Delta))$. We can then derive (dropping the subscripts $i+1$ and using the notation $\alpha_h = \alpha/\theta$):

$$\frac{w(\gamma) - w(\gamma - \Delta)}{\Delta} = \frac{\rho H}{\alpha(\gamma - \Delta)} - \frac{\alpha(\gamma) - \alpha(\gamma - \Delta)}{\Delta} \frac{\rho + \gamma}{\alpha(\gamma)\alpha(\gamma - \Delta)} \rho H$$
$$\frac{w(\gamma) - w(\gamma - \Delta)}{\Delta} = \frac{(\rho + \gamma)\rho Z_p^*}{\alpha_h(\gamma)\alpha_h(\gamma - \Delta)} \frac{\alpha_h(\gamma) - \alpha_h(\gamma - \Delta)}{\Delta}$$

Taking $\lim_{\Delta \to 0}$ and rearranging:

$$\frac{\partial w}{\partial \gamma} = \frac{\rho H}{\alpha}\left[1 - \varepsilon \frac{\frac{\partial \theta}{\partial \gamma}}{\theta}(\rho + \gamma)\right]$$
$$\frac{\partial w}{\partial \gamma} = -(\rho + \gamma)(1 - \varepsilon)\frac{\frac{\partial \theta}{\partial \gamma}}{\alpha}\rho Z_p^*$$

Solving for θ' and w':

$$\frac{\partial \theta}{\partial \gamma} = -\frac{1}{\rho+\gamma}\left[(1-\varepsilon)\frac{\rho Z_p^{*(i+1)}}{\rho H} - \frac{\varepsilon}{\theta}\right]^{-1}$$
$$\frac{\partial w}{\partial \gamma} = \frac{(1-\varepsilon)\rho Z_p^*}{\alpha}\left[(1-\varepsilon)\frac{\rho Z_p^*}{\rho H} - \frac{\varepsilon}{\theta}\right]^{-1}$$

Thus:

$$\frac{\partial w}{\partial \theta} = -(\rho+\gamma)\frac{1-\varepsilon}{\alpha}\rho Z_p^* < 0$$

θ_p^* is increasing in γ implies:

$$\rho Z_p^* > \frac{1}{\theta_p^*}\frac{\varepsilon}{1-\varepsilon}\rho H$$

$$\frac{\partial \theta_p^*}{\partial \gamma} < 0 \quad \forall \gamma < \gamma_I$$
$$\frac{\partial w_p^*}{\partial \gamma} > 0 \quad \forall \gamma < \gamma_I$$

They go to ∞ for $\gamma = \gamma_I$. $\partial w / \partial \theta$ at the border is well defined:

$$\frac{\partial w_p^*}{\partial \theta_p^*} = -(\rho+\gamma)\frac{\varepsilon}{\theta_p^{*I}\alpha(\theta_p^{*I})}\rho H \quad \text{for } \gamma = \gamma_I$$

Lastly in steady state the number of vacancies created must equal the number filled, so that: $\frac{v_p^{*i}}{\pi_i - u_p^{*i} - v_p^{*i}} = \frac{\gamma_i}{\alpha_l(\theta_p^{*i})}$ $\square$

Proof of Result 3

For any given $\tilde{\gamma} \in \Gamma$, define the constant $k \equiv Z_r^*(\tilde{\gamma}) - \tilde{Z}_p^*(\tilde{\gamma})$. Note that the function $Z_r^* - k = \tilde{Z}_p^*$ at $\tilde{\gamma}$ and $\frac{d(Z_r^* - k)}{d\gamma} = \frac{dZ_r^*}{d\gamma}$. Also, $\forall \Delta > 0$, $Z_r^*(\tilde{\gamma} - \Delta) - k > \tilde{Z}_p^*(\tilde{\gamma} - \Delta)$. So $\frac{d\tilde{Z}_p^*}{d\gamma} > \frac{dZ_r^*}{d\gamma} = \frac{dZ_o^*}{d\gamma}$ $\square$